COLLEGIATE MATHEMATICS FOR WAR SERVICE.

COMMUNICATIONS FOR THIS DEPARTMENT SHOULD BE SENT TO H. BLUMBERG, University of Illinois.

SOME DRAWINGS AND GRAPHICAL SOLUTIONS IN NAVIGATION.

By WM. H. ROEVER.

1. *Introductory Note.*—During the past summer the U. S. Naval Auxiliary Reserve School, Municipal Pier, Chicago, made arrangements whereby its students were to take a course in navigation at the University of Chicago. As a member of the teaching staff at the University during the summer, I offered my services as instructor in navigation to these students. Professor Wilczynski and I were asked to give the work in nautical astronomy. Most of the students, while very intelligent, had only a limited knowledge of mathematics,—not extending beyond plane trigonometry,—and no knowledge whatever of astronomy. For this reason I at first made frequent appeal to their geometric intuition and solved by graphical processes the spherical triangles which arise in the determination of time, latitude, longitude, azimuth, and in the problem of great-circle sailing. I also drew a number of pictures, illustrating space relations. These pictures and graphical processes proved to be so helpful to my students that it seemed worth while to those who had charge of the course in Navigation at the University of Chicago to have them put in a form whereby they will be available to others. Hence this article. It should not be assumed from what has been said, that I did not also give the customary methods of computation.

2. *Time Determination.*—The determination of time reduces itself to the mathematical problem of finding an angle (the hour angle) of a spherical triangle of which the three sides (complements of the latitude, altitude and declination) are known. To solve graphically such a triangle let us consider the trihedral formed by the planes of the sides of the spherical triangle. Fig. 1 represents such

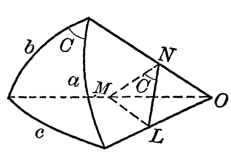

FIG. 1.

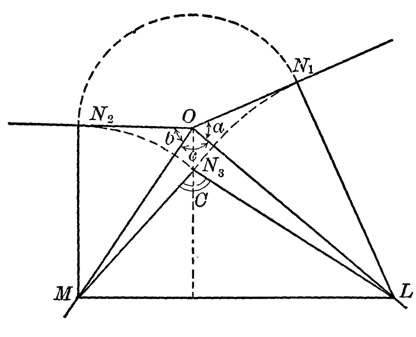

FIG. 2.

a trihedral in which a, b, c denote the sides of the spherical triangle or the face angles of the trihedral, and C denotes the required angle of the spherical triangle

or the corresponding dihedral angle of the trihedral. The point O represents the center of the sphere or the vertex of the trihedral, and N represents any point on the edge of the required dihedral angle. At N perpendiculars are erected to the edge ON in the faces which meet in ON. These perpendiculars meet the other edges of the trihedral in the points L and M. Then, in the triangle represented by LMN, the angle at N is equal to the required angle C of the spherical triangle. To find C graphically, let us cut the trihedral along the edge ON and open it up, or develop it, as shown in Fig. 2. Then N will assume the two positions N_1 and N_2, which lie at equal distances from O on the two positions assumed by the edge ON. The perpendiculars erected to these positions at the points N_1 and N_2, respectively, meet the new positions of the other edges in L and M, respectively. Hence ML, LN_1 and MN_2 in Fig. 2 are the true lengths of the sides of the triangle

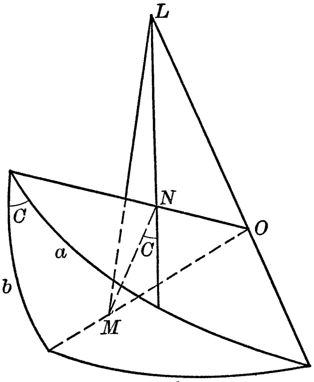

FIG. 3.

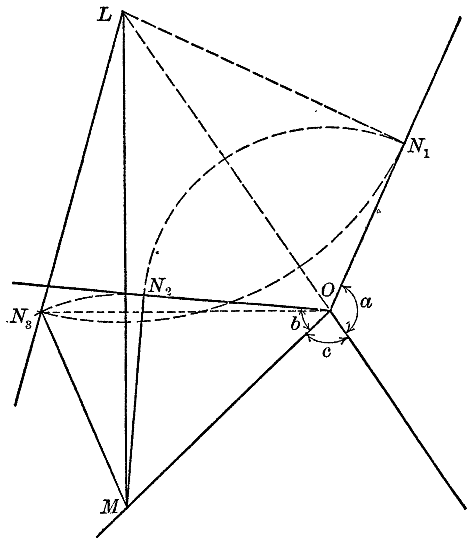

FIG. 4.

represented by LMN in Fig. 1. Thus we are led to the *Construction:* With O (Fig. 2) as vertex lay off the angle c equal to the side of the spherical triangle which lies opposite to the required angle C. Adjacent to the angle c and on either side of it lay off the angles a and b equal to the other two given sides of the spherical triangle. Then on the terminal lines of these last two angles take points N_1 and N_2 at equal distances from O. At these points erect perpendiculars N_1L and N_2M to these sides, meeting the sides of the angle c in the points L and M, respectively. With L as center and LN_1 as radius draw a circle, and with M as center and MN_2 as radius draw another circle. These two circles intersect in a point N_3, which, together with the points L and M forms the triangle represented by LMN in Fig. 1, and hence the angle LN_3M of Fig. 2 is the required angle C of the given spherical triangle. A check on the construction is furnished by the fact that the points N_3 and O must lie on the same perpendicular to the line LM.

In navigation, the sides b and c (complements of the latitude and altitude, respectively) are always acute, but the side a (complement of the declination) may be acute or obtuse. For the case in which the side a is obtuse the point L will lie on the produced edge, as shown in Fig. 3, and then the required angle C is the supplement of the angle MNL of the triangle MLN. The corresponding construction is shown in Fig. 4.

If the sides a and b of the given spherical triangle are nearly 90°, the points L and M of the constructions shown in Figs. 2 and 4, will fall beyond the limits of the drawings. In this case the construction explained in § 4 can be used to advantage.

3. *Solution of the triangle in the Saint Hilaire Method.*—To find the latitude and longitude of a ship by the Sumner Method, two *lines of position* must be determined. The Saint Hilaire Method for finding such lines, involves a spherical triangle of which two sides and the included angle are known and of which the side opposite the given angle is required.[1] To solve graphically this spherical triangle, let us consider Fig. 5, in which the edge of the given dihedral angle C

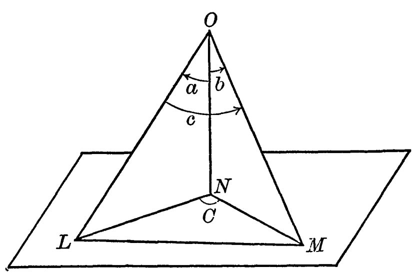

Fig. 5.

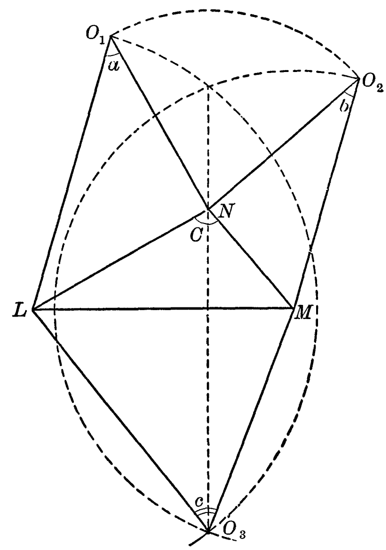

Fig. 6.

of the corresponding trihedral is represented as being in a vertical position, *i. e.*, perpendicular to the plane (let us say) of a table top. The points in which the edges of the trihedral meet this plane are represented by the points N, L, and M, of which the first lies on the edge of the given dihedral angle C. The center of the sphere, or the vertex of the trihedral, is represented by the point O. Let us now think of this trihedral as cut along its three edges ON, OM, OL and

[1] This side is the computed zenith distance of the observed object. The azimuth is usually obtained from an azimuth table, but it can easily be found graphically as shown in the *Remark* at the nd of § 4.

of its faces as turned down around the lines NM, ML, LN into the plane of the table top. They then assume the position shown in Fig. 6. Hence the *Construction:* With N (Fig. 6) as vertex lay off the angle C equal to the given angle of the spherical triangle. At N erect perpendiculars to the sides of this angle and on these take points O_1 and O_2 at equal distances from N. Through O_1 and O_2 draw lines making with O_1N and O_2N the angles a and b, respectively, equal to the given sides of the spherical triangle, and meeting the sides of the angle C in the points L and M, respectively. Then with L as center and LO_1 as radius draw a circle, and with M as center and MO_2 as radius draw another circle. These circles meet in a point O_3, which with L and M forms the triangle represented in

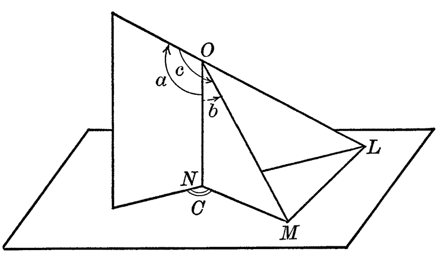

FIG. 7.

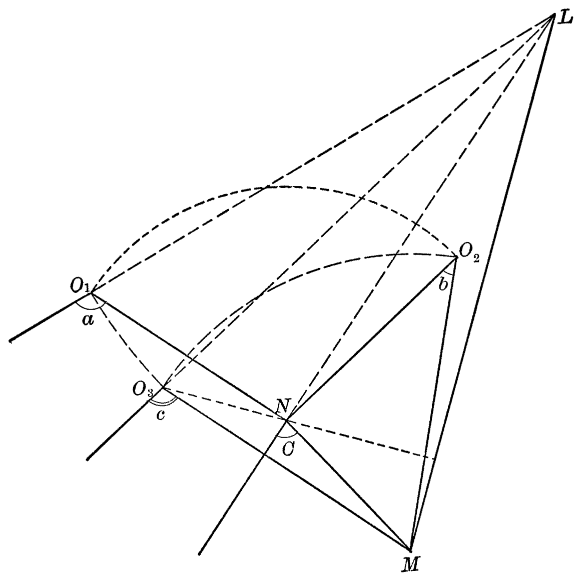

FIG. 8.

Fig. 5 by OLM and of which the angle at O is equal to the required side c of the spherical triangle. Hence the angle LO_3M of Fig. 6 is the required angle c of the spherical triangle. A check is furnished by the fact that O_3 and N should lie on the same perpendicular to LM.

In the navigation problem the side b (complement of the latitude) is always acute, but the parts C and a may be acute or obtuse. In the case where the part a is obtuse the point L lies on the produced edge, as shown in Fig. 7, and then the required part c is the supplement of the angle LOM in the triangle LOM. The corresponding construction is shown in Fig. 8.

If the parts a and b are nearly 90° the points L and M of Figs. 6 and 8 will fall beyond the limits of the drawings, and in this case the construction of § 4 can be used.

4. *Constructions available in failing cases of the above.*—Constructions which have the advantage over the constructions just given, in that all the points re-

main within the limits of the drawings, will now be given. These constructions are perfectly general and may be used whenever the given parts of the spherical triangle are less then 180°. These constructions will now be given, but for their proof the reader is referred to *Loria, Vorlesungen ueber Darstellende Geometrie*, Vol. II, p. 6.

Construction (Fig. 9 for a acute, Fig. 10 for a obtuse).—The three parts a, b, c (*i. e.*, the sides of the spherical triangle or the face angles of the corresponding trihedral) are laid off from a point O as shown in Figs. 9 and 10. A circle Γ of center O and any convenient radius is then drawn cutting the sides of the angles a, b, c in the points $\bar{C}$, $\bar{A}$, $\bar{B}_1$ and $\bar{B}_2$ as shown in the figures. From the points $\bar{B}_1$ and $\bar{B}_2$ perpendiculars are dropped to the rays $O\bar{C}$ and $O\bar{A}$, respectively,

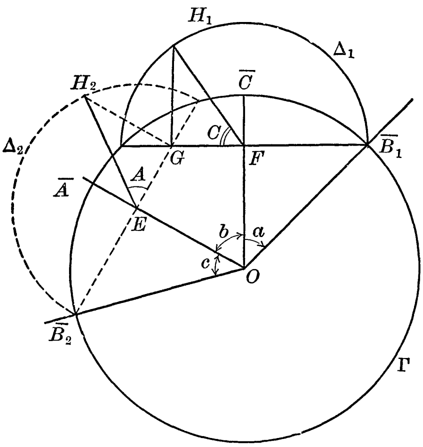

FIG. 9

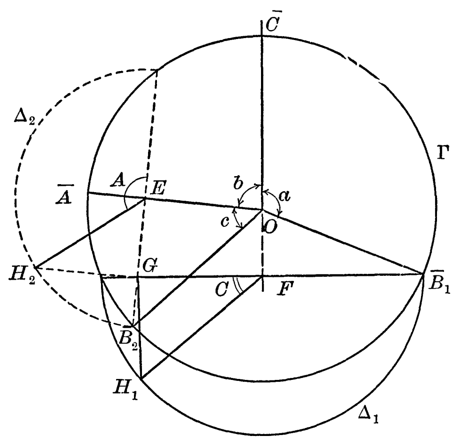

FIG. 10.

cutting these in the points F and E, respectively, and each other in the point G. With F as center and $F\bar{B}_1$ as radius a circle Δ_1 is drawn. At G a perpendicular is erected to the line $\bar{B}_1F$ cutting the circle Δ_1 in H_1. Then the supplement of the angle $\bar{B}_1FH_1$ is the required angle C of the spherical triangle of which the sides are the angles a, b, c.

Construction (Fig. 9 for a acute, Fig. 10 for a obtuse).—If a, b, C are the given parts of the spherical triangle, the circle Γ, the perpendicular $\bar{B}_1F$ and the circle Δ_1 are drawn as in the above preceding construction. The point H_1 is now taken on the circle Δ_1, so that the angle $\bar{B}_1FH_1$ is the supplement of the given angle C. From H_1 a perpendicular is dropped to $F\bar{B}_1$, meeting it in G. From G a perpendicular is dropped to $O\bar{A}$ meeting it in E, and meeting the circle Γ in $\bar{B}_2$. The angle $\bar{A}O\bar{B}_2$ is then equal to the required side c of this spherical triangle.

Remark.—In order to find the angle A (which is the azimuth in the nautical problem), we have merely to draw the circle Δ_2 of center E and radius $E\bar{B}_2$. At G we then erect a perpendicular to $\bar{B}_2E$ cutting Δ_2 in H_2. The supplement of the angle $\bar{B}_2EH_2$ is then equal to the required angle A.

5. *The great-circle-sailing problem.*—The problem of great-circle sailing consists in determining the latitude and longitude of various points Q of the great circle which connects two given points A and B on the surface of the earth. The points Q are then plotted on a Mercator chart[1] from which the course of the ship between consecutive points is determined. To devise a graphical solution of this problem let us first think of the points A and B as both situated in the same hemisphere (Fig. 11).

Now think of a plane tangent to the earth at its pole O' and upon this plane let us project the points A and B from the center O of the earth, denoting the projections by A' and B', respectively. Then the straight line $A'B'$ of this plane is the projection of the great circle path AB, and the lines $O'A'$ and $O'B'$ are the

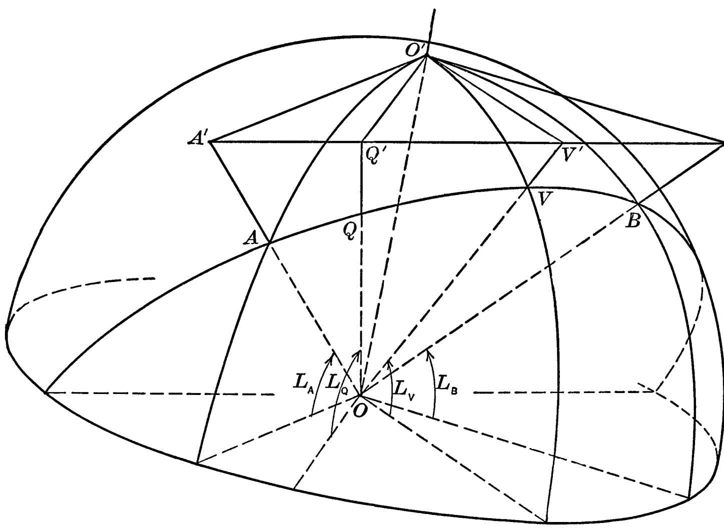

FIG. 11.

projections of the meridians of A and B. Let V denote the point of the great circle AB which is nearest the pole O'. Evidently the projection of the point V is the foot V' of the perpendicular from O' to $A'B'$ in the triangle $O'A'B'$. A general point Q of the great circle AB projects into a point Q' of the line $A'B'$. The tetrahedron $OA'B'O'$ represented in Fig. 11 is the inverted position of the tetrahedron $OLMN$ represented in Fig. 5, so that the parts a, b, C of Fig. 5 correspond respectively to the co-latitude ($90°$-L_A) of A, the co-latitude ($90°$-L_B) of B and the difference in longitude (equal to the angle $B'O'A'$) of the points A and B. Hence the part c of Fig. 5 corresponds to the angular distance AOB of Fig. 11. Thus the construction of Fig. 6 determines graphically (in that it determines the part c) the angular distance between the two points A and B when the latitudes and the difference in longitudes of these points are known. In order to find the latitude, longitude and distance from A (or from B) of the

[1] For the definition of Mercator projection see § 7.

point V and of the general point Q, we will begin by repeating the construction of Fig. 6. In Fig. 12 this much is shown by the heavily drawn lines; the points O', A', B', O_A, O_B, O''' of Fig. 12 correspond to the points N, L, M, O_1, O_2, O_3, respectively, of Fig. 6. In Fig. 12 the triangle $O'A'B'$ is the true shape of the triangle which is shown in Fig. 11 by the same letters. Hence in Fig. 12 the foot of the perpendicular from O' to $A'B'$ is the point V', and any point of $A'B'$ is a point Q'. The angles $A'O'''V'$ and $A'O'''Q'$ are then the angular distances of the points V and Q, respectively, from the point A. Evidently the angle $V'O'A'$

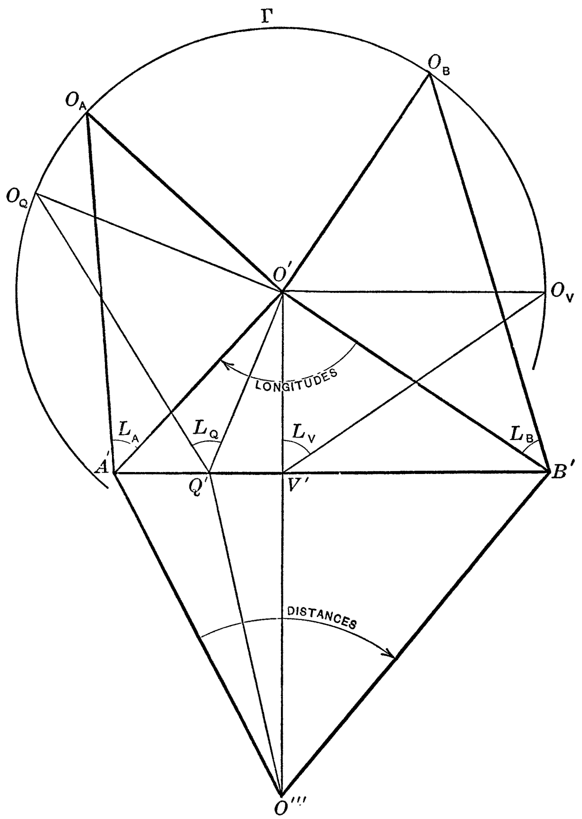

FIG. 12.

(Fig. 12) is the difference in longitude between the points V and A, and the angle $V'O'Q'$ is the difference in longitude between V and Q. To find the latitudes of V and Q let us first draw a circle Γ of center O' and radius $O'O_A = O'O_B$ (equal to the distance represented by OO' in Fig. 11). To get the latitude of V draw in Fig. 12 a line through O' perpendicular to $O'V'$ and cutting Γ in O_v. Then the angle $O'V'O_v$ is the latitude of V. Similarly, to get the latitude of Q draw a line through O' perpendicular to $O'Q'$ and cutting Γ in O_Q. The angle $O'Q'O_Q$ is then the latitude of Q. This is evident from the fact that the triangles $O'O_vV'$ and

$O'O_QQ'$ of Fig. 12 are the revolved positions (turned into the plane of the triangle $O'A'B'$) of the triangles represented by $O'V'O$ and $O'Q'O$ in Fig. 11.

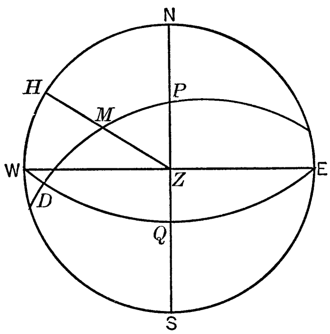

Fig. 13.

Just as we started with Fig. 6 the construction of Fig. 12 corresponding to the case where A and B are in the same hemisphere, so for the case where A and $B$are in different hemispheres we start the construction with Fig. 8.

6. *Definition of a good picture.*—In the absence of models, good pictures serve very well to convey an adequate notion of space relations. By a good picture of a space object is meant a plane representative which, when properly placed, produces upon the retinal surface of the eye an image which does not differ much from that produced by the object itself in a position in which one is accustomed to see or think of the object. It can easily be shown that a central

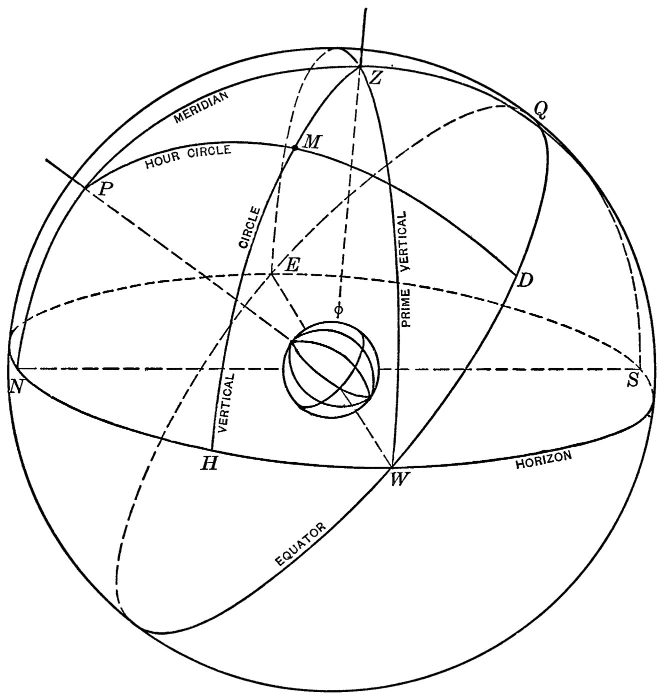

Fig. 14.

or parallel (orthographic or oblique) projection in general satisfies this criterion[1] and therefore such a projection of an object in an accustomed position furnishes a

[1] See for instance, pages 148 and 152 of the article on Descriptive Geometry in Vol. 25 of this Monthly.

good picture. What is here meant by an accustomed position will be made clear from the statements which follow. A stereographic projection of a sphere is a central projection for a particular position of the center of projection.[1] However, the stereographic projection of the celestial sphere with its circles of reference does

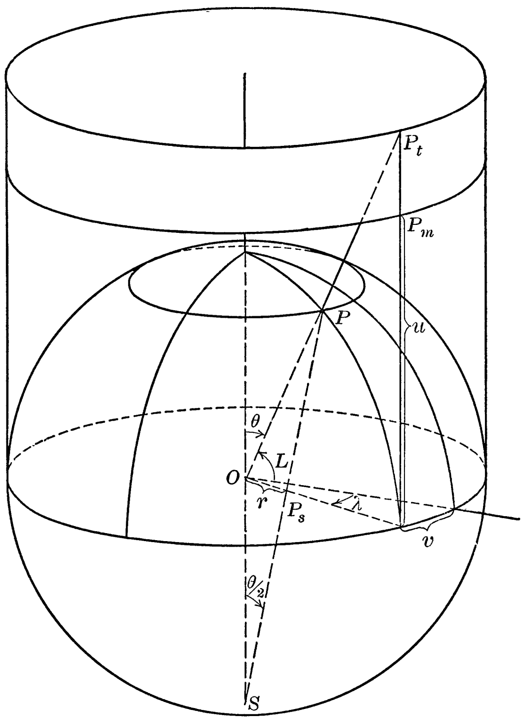

FIG. 15.

not produce nearly so good a picture as does an orthographic projection on a general plane. This will be evident to the reader when a comparison is made of Figs. 13 and 14 which are stereographic and orthographic projections, respectively. Hence a stereographic projection, although it is a projection which satisfies the criterion of producing the same retinal image as the object itself, does not furnish a good picture, because it is projected from a point from which one is not accustomed to seeing the sphere.

7. *Stereographic and Mercator projections defined.*—We have already used the terms Mercator projection and stereographic projection. For the purpose of defining these, as well as showing a simple relation which exists between them, we will make use of the foregoing picture (Fig. 15) which is an orthographic projection.

The stereographic projection of a sphere is the central projection of the points of the surface of the sphere upon a diametral plane from a pole of the great circle

[1] For the definition of stereographic projection see § 7.

in which this plane cuts the sphere. In Fig. 15 the plane of projection is the equatorial plane, and the center of projection is the south pole. The projection of the point P of the sphere is the point P_s of the equatorial plane.

The Mercator projection of a sphere is the development of a right circular cylinder, tangent to the sphere along the equator, the points of which are obtained from those of the sphere in the following manner. The point P of the sphere is projected from the center O into a point P_t of the cylinder (Fig. 15). Upon the element of the cylinder which passes through the point P_t the Mercator projection P_m of the point P lies, and its distance u from the equator is equal to the Naperian logarithm of the reciprocal $(1/r)$ of the distance (r) of the stereographic projection P_s from the center O of the sphere, the radius of which is taken to be unity.

Thus if of a point P of the sphere, L = latitude and λ = longitude, and θ is the co-latitude, so that

$$\theta = \frac{\pi}{2} - L \text{ and hence } \theta/2 = \pi/4 - L/2,$$

and if for the stereographic projection P_s the polar coördinates are r and λ, and for the Mercator projection the rectangular coördinates are u and v, then

$$\begin{cases} r = \tan\dfrac{\theta}{2} = \tan\left(\dfrac{\pi}{4} - \dfrac{L}{2}\right), \\ \lambda = \lambda, \end{cases}$$

and

$$\begin{cases} u = \log_e \dfrac{1}{r} = -\log_e \tan\dfrac{\theta}{2} = -\log_e \tan\left(\dfrac{\pi}{4} - \dfrac{L}{2}\right), \\ v = \lambda. \end{cases}$$

Thus one easily sees the simple relation which exists between these two types of map.

8. *Mean sun and equation of time defined.*—In the definitions of mean time and the equation of time, a good picture showing the relation which exists between the mean and apparent suns is very helpful. In the orthographic projection shown in Fig. 16 the celestial sphere with the equinoctial and ecliptic are represented. In this figure, the vernal equinox is represented by the point r, the position which the true sun occupies at perihelion (which occurs about January first) is represented by S_0, and the position which the true sun occupies at any particular instant (the position shown in the figure corresponds to about February 12) is represented by S. Before defining the mean sun, let us first define a fictitious sun which moves in the ecliptic at a uniform rate in the same direction as the true sun does and coincides with the true sun at perihelion (and therefore also at aphelion). The point F represents the position which this sun occupies when the true sun is at S. The *mean* sun may now be defined as the point which moves along the equinoctial at a uniform rate and coincides with the fictitious sun at the vernal equinox (and therefore also at the autumnal equinox).

The position which the mean sun occupies when the true sun is at S is represented by M. In other words, the right ascension of the mean sun M is equal to the mean celestial longitude of the fictitious sun F, these two artificial suns being in coincidence at the vernal equinox. Fig. 16 also represents the center O and the north pole P of the celestial sphere and the hour circles of S, F and M. The hour circles of S and F intersect the equinoctial in the points

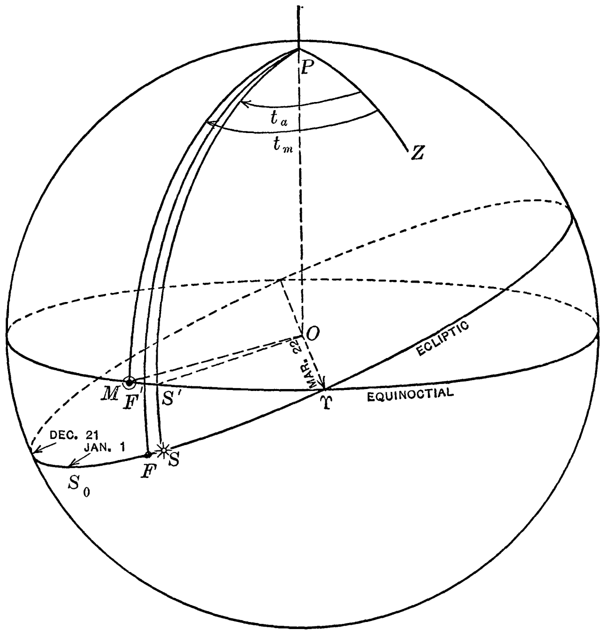

FIG. 16.

S' and F', respectively. The difference in right ascension between S and M, *i. e.*, the angle MOS' is called the equation of time. It is the difference between mean and apparent solar times. Thus if Z represents the zenith of a particular place of the earth at a particular instant,

$$t_m, = \text{angle } ZPM, \text{ is the local mean time of this place,}$$
$$t_a, = \text{angle } ZPS, \text{ is the local apparent solar time of this place,}$$

and

$$\epsilon, = t_m - t_a = \text{angle } MOS', \text{ is the equation of time.}^1$$

9. *The Sumner Method.*—In describing the Sumner Method of determining the position of a ship at sea, a picture representing both the celestial and the terrestrial spheres proves to be very helpful. In Fig. 17 such a picture is represented in orthographic projection. The positions on the celestial sphere, of the pole, zenith and two positions of the sun are represented by P, Z, S_1, S_2, respectively. The positions on the terrestrial sphere, of the points immediately below

[1] It should not be forgotten that hour angle (time) and right ascension are measured in opposite directions.

these, are denoted by p, z, s_1, s_2, respectively. Then z is the position of the ship. and s_1, s_2 are called the subsolar (or in the case of a star, the substellar) points, The meridian of Greenwich is shown on both of the spheres as is also the equator. It is thus evident that the terrestrial longitude of a subsolar point is equal to the Greenwich apparent time and that the latitude is equal to the declination of the true sun. A small circle of the terrestrial sphere which passes through the position of the ship (z) and has as pole the subsolar point is called a Sumner circle. It is

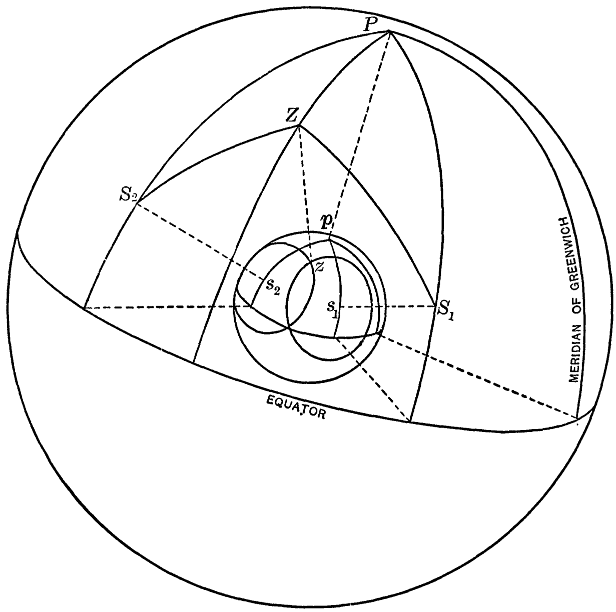

FIG. 17.

easy to see from the figure that the angular radius of this circle is equal to the zenith distance of the sun. The figure shows the two Sumner circles which correspond to S_1 and S_2 and which intersect in the position z of the ship.

10. *The Gnomonic Chart.*—By a gnomonic chart is meant the projection of the points of a spherical surface upon a tangent plane to this surface from the center of the sphere. A straight line of such a chart corresponds to a great circle of the sphere. Such charts are used in solving graphically the problem of great-circle sailing, of which one graphical solution has already been given in § 5. In fact the projection $O'A'B'$ (Fig. 11) on the plane tangent to the sphere at the point O' is a gnomonic chart of the triangle $O'AB$ of the sphere for the case where the plane of projection is tangent to the sphere at the pole O'. In this case the meridians project into straight lines radiating from the point of tangency and the parallels of latitude project into concentric circles whose common center is the point of tangency. If, however, the plane of projection is tangent to the sphere at a point not coincident with a pole, the meridians project into straight lines which

radiate from the point where the axis of the earth pierces the plane of projection, and the parallels of latitude project into conics. This is the case because the planes of the meridians all pass through the axis, which contains the center of projection, and because the projecting rays of a parallel of latitude form a right circular cone. Fig. 18 enables one to see this more clearly. If on such a chart a sufficient number of meridians and parallels of latitude are represented, it is an easy matter to read off from the chart the latitude and longitude of various points of the straight line representative of a great-circle path of the sphere. The method of doing this as well as that of finding the distance and course (direction of sailing) is explained on the great-circle charts which are published by the U. S. Hydrographic Office.

The construction of a gnomonic chart is identical with that of a horizontal sun-dial for a place whose latitude is equal to that of the point of tangency of the plane of projection of this chart. This can be easily seen by the aid of Fig. 18, in which we may now consider the horizontal table top as the dial plate and the axis of the sphere as the gnomon, or style, of the dial. The lines representing the meridians are then positions of the shadow of the style upon the dial plate, and the conics representing the parallels of latitude are the diurnal paths of the

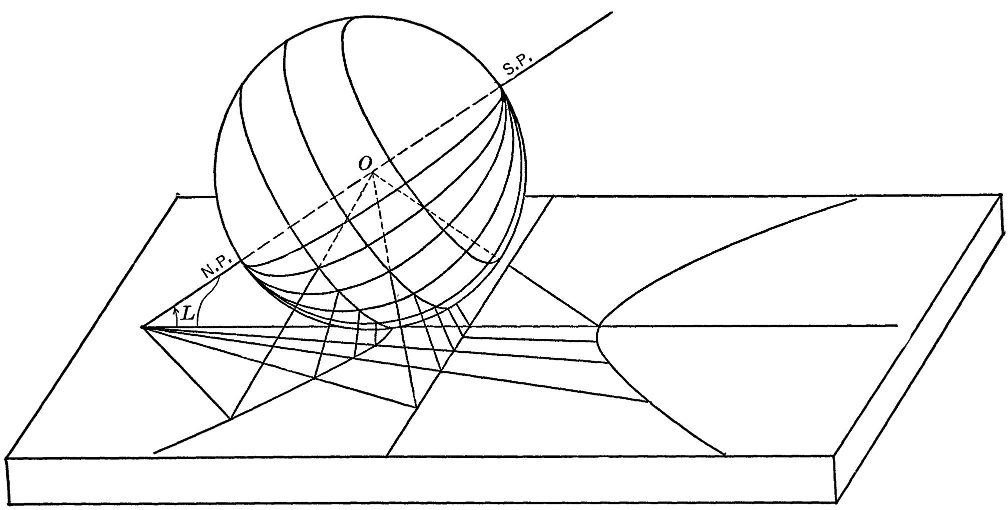

FIG. 18.

shadow of a knob (situated at the point O) of the style on the dial plate. In the dialing problem it is required to find the position of the shadow of the style for the hours of the day. The diurnal paths of the knob are the conics lying between the conics representing the Tropics of Capricorn and Cancer. It is on these conics that the equation of time is laid off from the position of the shadow of the style at apparent solar noon in order to make possible the use of a sundial for the determination of mean time.